5-2 Verifying Trigonometric Identities

Verify each identity.

1. $(\sec^2\theta - 1)\cos^2\theta = \sin^2\theta$

SOLUTION:

$(\sec^2\theta - 1)\cos^2\theta$
$= (\tan^2\theta)\cos^2\theta$ Pythagorean Identity
$= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta$ Quotient Identity
$= \sin^2\theta$ Multiply and divide out common factor.

2. $\sec^2\theta\,(1 - \cos^2\theta) = \tan^2\theta$

SOLUTION:

$\sec^2\theta(1 - \cos^2\theta)$
$= \sec^2\theta - \sec^2\theta\cos^2\theta$ Distributive Property
$= \sec^2\theta - \frac{1}{\cos^2\theta}\cdot\cos^2\theta$ Reciprocal Identity
$= \sec^2\theta - 1$ Multiply and divide out common factor.
$= \tan^2\theta$ Pythagorean Identity

3. $\sin\theta - \sin\theta\cos^2\theta = \sin^3\theta$

SOLUTION:

$\sin\theta - \sin\theta\cos^2\theta$
$= \sin\theta(1 - \cos^2\theta)$ Factor.
$= \sin\theta\sin^2\theta$ Pythagorean Identity
$= \sin^3\theta$ Multiply.

4. $\csc\theta - \cos\theta\cot\theta = \sin\theta$

SOLUTION:

$\csc\theta - \cos\theta\cot\theta$
$= \frac{1}{\sin\theta} - \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)$ Reciprocal/Quotient Identities
$= \frac{1 - \cos^2\theta}{\sin\theta}$ Write with a com denom
$= \frac{\sin^2\theta}{\sin\theta}$ Pythagorean Identity
$= \sin\theta$ Divide out factor of $\sin\theta$

5. $\cot^2\theta\csc^2\theta - \cot^2\theta = \cot^4\theta$

SOLUTION:

$\cot^2\theta\csc^2\theta - \cot^2\theta$
$= \cot^2\theta(\csc^2\theta - 1)$ Factor.
$= \cot^2\theta\cot^2\theta$ Pythagorean Identity
$= \cot^4\theta$ Multiply and add exponents.

6. $\tan\theta\csc^2\theta - \tan\theta = \cot\theta$

SOLUTION:

$\tan\theta\csc^2\theta - \tan\theta$
$= \tan\theta(\csc^2\theta - 1)$ Factor
$= \tan\theta\cot^2\theta$ Pythagorean Identity
$= \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}$ Quotient Identities
$= \frac{\cos\theta}{\sin\theta}$ Multiply and divide common factors.
$= \cot\theta$ Quotient Identity

7. $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$

SOLUTION:

$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$
$= \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$ Reciprocal Identity
$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$ Common denominator
$= \frac{1 - \sin^2\theta}{\sin\theta\cos\theta}$ Write with a com denom
$= \frac{\cos^2\theta}{\sin\theta\cos\theta}$ Pythagoren Identity
$= \frac{\cos\theta}{\sin\theta}$ Divide out $\cos\theta$.
$= \cot\theta$ Quotient Identity

8. $\frac{\sin\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{\sin\theta} = 2\csc\theta$

SOLUTION:

$\frac{\sin\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{\sin\theta}$
$= \frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{1 - \cos\theta}\cdot\frac{1 - \cos\theta}{\sin\theta}$ Rewrite using com denom
$= \frac{\sin^2\theta}{\sin\theta(1 - \cos\theta)} + \frac{1 - 2\cos\theta + \cos^2\theta}{\sin\theta(1 - \cos\theta)}$ Mulitply
$= \frac{\sin^2\theta + \cos^2\theta + 1 - 2\cos\theta}{\sin\theta(1 - \cos\theta)}$ Write with a com denom
$= \frac{1 + 1 - 2\cos\theta}{\sin\theta(1 - \cos\theta)}$ Pythagorean Identity
$= \frac{2 - 2\cos\theta}{\sin\theta(1 - \cos\theta)}$ Add
$= \frac{2(1 - \cos\theta)}{\sin\theta(1 - \cos\theta)}$ Factor
$= \frac{2}{\sin\theta}$ Divide out $(1 - \cos\theta)$
$= 2\csc\theta$ Reciprocal Identity

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9. $\dfrac{\cos\theta}{1+\sin\theta} + \tan\theta = \sec\theta$

SOLUTION:

$\dfrac{\cos\theta}{1+\sin\theta} + \tan\theta$

$= \dfrac{\cos\theta}{1+\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$ Quotient Identity

$= \dfrac{\cos\theta}{\cos\theta}\cdot\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{1+\sin\theta}\cdot\dfrac{\sin\theta}{\cos\theta}$ Rewrite 1 with com denom

$= \dfrac{\cos^2\theta}{\cos\theta(1+\sin\theta)} + \dfrac{\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}$ Multiply

$= \dfrac{\cos^2\theta+\sin\theta+\sin^2\theta}{\cos\theta(1+\sin\theta)}$ Write as a single fraction

$= \dfrac{1+\sin\theta}{\cos\theta(1+\sin\theta)}$ Pythagorean Identity

$= \dfrac{1}{\cos\theta}$ Divide out $(1+\sin\theta)$

$= \sec\theta$ Reciprocal Identity

10. $\dfrac{\sin\theta}{1-\cot\theta} + \dfrac{\cos\theta}{1-\tan\theta} = \sin\theta + \cos\theta$

SOLUTION:

$\dfrac{\sin\theta}{1-\cot\theta} + \dfrac{\cos\theta}{1-\tan\theta}$

$= \dfrac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}} + \dfrac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}$ Quotient Identity

$= \dfrac{\sin\theta}{\frac{\sin\theta}{\sin\theta}-\frac{\cos\theta}{\sin\theta}} + \dfrac{\cos\theta}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}$ Rewrite using com denom

$= \dfrac{\sin\theta}{\frac{\sin\theta-\cos\theta}{\sin\theta}} + \dfrac{\cos\theta}{\frac{\cos\theta-\sin\theta}{\cos\theta}}$ Write denom as single fractions

$= \dfrac{\sin^2\theta}{\sin\theta-\cos\theta} + \dfrac{\cos^2\theta}{\cos\theta-\sin\theta}$ Simplify fractions

$= \dfrac{\sin^2\theta}{\sin\theta-\cos\theta} - \dfrac{\cos^2\theta}{\sin\theta-\cos\theta}$ Factor out -1

$= \dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$ Write as a single fraction

$= \dfrac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta}$ Factor numerator

$= \sin\theta+\cos\theta$ Divide out $(\sin\theta-\cos\theta)$

11. $\dfrac{1}{1-\tan^2\theta} + \dfrac{1}{1-\cot^2\theta} = 1$

SOLUTION:

$\dfrac{1}{1-\tan^2\theta} + \dfrac{1}{1-\cot^2\theta}$

$= \dfrac{1}{1-\frac{\sin^2\theta}{\cos^2\theta}} + \dfrac{1}{1-\frac{\cos^2\theta}{\sin^2\theta}}$ Quotient Identity

$= \dfrac{1}{\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}} + \dfrac{1}{\frac{\sin^2\theta}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}}$ Rewrite 1 using com denom

$= \dfrac{1}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}} + \dfrac{1}{\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta}}$ Write denom as single fractions

$= \dfrac{\cos^2\theta}{\cos^2\theta-\sin^2\theta} + \dfrac{\sin^2\theta}{\sin^2\theta-\cos^2\theta}$ Simplify fractions

$= \dfrac{\cos^2\theta}{\cos^2\theta-\sin^2\theta} + \dfrac{-\sin^2\theta}{\cos^2\theta-\sin^2\theta}$ Common denominator

$= \dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta-\sin^2\theta}$ Write as a single fraction

$= 1$ Divide out $(\cos^2\theta-\sin^2\theta)$

12. $\dfrac{1}{\csc\theta+1} + \dfrac{1}{\csc\theta-1} = 2\sec^2\theta\sin\theta$

SOLUTION:

$\dfrac{1}{\csc\theta+1} + \dfrac{1}{\csc\theta-1}$

$= \dfrac{\csc\theta-1}{\csc\theta-1}\cdot\dfrac{1}{\csc\theta+1} + \dfrac{\csc\theta+1}{\csc\theta+1}\cdot\dfrac{1}{\csc\theta-1}$ Common denominator

$= \dfrac{\csc\theta-1}{\csc^2\theta-1} + \dfrac{\csc\theta+1}{\csc^2\theta-1}$ Multiply

$= \dfrac{2\csc\theta}{\csc^2\theta-1}$ Write as a single fraction

$= \dfrac{2\csc\theta}{\cot^2\theta}$ Pythagorean Identity

$= \dfrac{2\left(\frac{1}{\sin\theta}\right)}{\frac{\cos^2\theta}{\sin^2\theta}}$ Reciprocal/Quotient Identities

$= \dfrac{2}{\sin\theta}\cdot\dfrac{\sin^2\theta}{\cos^2\theta}$ Rewrite using multiplication

$= \dfrac{2\sin\theta}{\cos^2\theta}$ Multiply

$= \left(\dfrac{2}{\cos^2\theta}\right)\sin\theta$ Factor

$= 2\sec^2\theta\sin\theta$ Reciprocal Identity

13. $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) = 1$

SOLUTION:

$(\csc\theta-\cot\theta)(\csc\theta+\cot\theta)$

$= \csc^2\theta-\cot^2\theta$ Multiply.

$= 1$ Pythagorean Identity

14. $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

SOLUTION:

$\cos^4\theta-\sin^4\theta$

$= (\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)$ Factor

$= 1(\cos^2\theta-\sin^2\theta)$ Pythagorean Identity

$= \cos^2\theta-\sin^2\theta$ Multiply

15. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$

SOLUTION:

$\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta}$

$= \dfrac{1+\sin\theta}{1+\sin\theta}\cdot\dfrac{1}{1-\sin\theta} + \dfrac{1-\sin\theta}{1-\sin\theta}\cdot\dfrac{1}{1+\sin\theta}$ Common denominator

$= \dfrac{1+\sin\theta}{1-\sin^2\theta} + \dfrac{1-\sin\theta}{1-\sin^2\theta}$ Multiply

$= \dfrac{2}{1-\sin^2\theta}$ Write as a single fraction

$= \dfrac{2}{\cos^2\theta}$ Pythagorean Identity

$= 2\sec^2\theta$ Reciprocal Identity

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16. $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{\cos\theta}{1-\sin\theta}=2\sec\theta$

SOLUTION:

$\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{\cos\theta}{1-\sin\theta}$

$=\dfrac{\cos\theta}{1+\sin\theta}\cdot\dfrac{1-\sin\theta}{1-\sin\theta}+\dfrac{\cos\theta}{1-\sin\theta}\cdot\dfrac{1+\sin\theta}{1+\sin\theta}$ Common denominator

$=\dfrac{\cos\theta(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}+\dfrac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$ Multiply

$=\dfrac{\cos\theta(1-\sin\theta)+\cos\theta(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$ Write as a single fraction

$=\dfrac{\cos\theta-\sin\theta\cos\theta+\cos\theta+\sin\theta\cos\theta}{1-\sin^2\theta}$ Multiply

$=\dfrac{2\cos\theta}{1-\sin^2\theta}$ Simplify the numerator

$=\dfrac{2\cos\theta}{\cos^2\theta}$ Pythagorean Identity

$=\dfrac{2}{\cos\theta}$ Divide out $\cos\theta$

$=2\sec\theta$ Quotient Identity

17. $\csc^4\theta-\cot^4\theta=2\cot^2\theta+1$

SOLUTION:

$\csc^4\theta-\cot^4\theta$

$=(\csc^2\theta-\cot^2\theta)(\csc^2\theta+\cot^2\theta)$ Factor

$=[\csc^2\theta-(\csc^2\theta-1)][\csc^2\theta+(\csc^2\theta-1)]$ Pythagorean Identity

$=[\csc^2\theta-\csc^2\theta+1][\csc^2\theta+\csc^2\theta-1]$ Multiply

$=[1][2\csc^2\theta-1]$ Add

$=2\csc^2\theta-1$ Multiply

$=2(\cot^2\theta+1)-1$ Pythagorean Identity

$=2\cot^2\theta+2-1$ Multiply

$=2\cot^2\theta+1$ Add

18. $\dfrac{\csc^2\theta+2\csc\theta-3}{\csc^2\theta-1}=\dfrac{\csc\theta+3}{\csc\theta+1}$

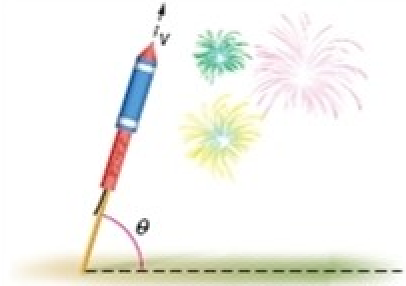
SOLUTION:

$\dfrac{\csc^2\theta+2\csc\theta-3}{\csc^2\theta-1}$

$=\dfrac{(\csc\theta+3)(\csc\theta-1)}{(\csc\theta+1)(\csc\theta-1)}$ Factor

$=\dfrac{\csc\theta+3}{\csc\theta+1}$ Divide out $(\csc\theta-1)$

19. **FIREWORKS** If a rocket is launched from ground level, the maximum height that it reaches is given by $h=\dfrac{v^2\sin^2\theta}{2g}$, where θ is the angle between the ground and the initial path of the rocket, v is the rocket's initial speed, and g is the acceleration due to gravity, 9.8 meters per second squared.



a. Verify that $\dfrac{v^2\sin^2\theta}{2g}=\dfrac{v^2\tan^2\theta}{2g\sec^2\theta}$.

b. Suppose a second rocket is fired at an angle of 80° from the ground with an initial speed of 110 meters per second. Find the maximum height of the rocket.

SOLUTION:

a.

$\dfrac{v^2\tan^2\theta}{2g\sec^2\theta}=\dfrac{v^2\left(\dfrac{\sin^2\theta}{\cos^2\theta}\right)}{2g\left(\dfrac{1}{\cos^2\theta}\right)}$ Quotient/Reciprocal Identities

$=\dfrac{v^2\sin^2\theta}{2g}$ Divide out $\dfrac{1}{\cos^2\theta}$

b. Evaluate the expression $\dfrac{v^2\sin^2\theta}{2g}$ for v = 110 m, $\theta=80°$, and g = 9.8 m/s^2.

$\dfrac{v^2\sin^2\theta}{2g}=\dfrac{110^2\sin^2 80°}{2(9.8)}$

≈ 598.7

The maximum height of the rocket is about 598.7 meters.

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Verify each identity.

20. (csc θ + cot θ)(1 – cos θ) = sin θ

SOLUTION:

$(\csc\theta + \cot\theta)(1-\cos\theta)$

$= \csc\theta - \csc\theta\cos\theta + \cot\theta - \cot\theta\cos\theta$ Multiply binomials

$= \frac{1}{\sin\theta} - \left(\frac{1}{\sin\theta}\right)\cos\theta + \left(\frac{\cos\theta}{\sin\theta}\right) - \left(\frac{\cos\theta}{\sin\theta}\right)\cos\theta$ Reciprocal/Quotient Identities

$= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin\theta}$ Multiply

$= \frac{1-\cos\theta+\cos\theta-\cos^2\theta}{\sin\theta}$ Write as one fraction with a common denominator.

$= \frac{1-\cos^2\theta}{\sin\theta}$ Simplify numerator

$= \frac{\sin^2\theta}{\sin\theta}$ Pythagorean Identity

$= \sin\theta$ Divide out common factor

21. $\sin^2\theta\tan^2\theta = \tan^2\theta - \sin^2\theta$

SOLUTION:

$\tan^2\theta - \sin^2\theta$

$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta$ Quotient Identity

$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \cdot 1$ Multiply by 1

$= \frac{\sin^2\theta}{\cos^2\theta} - \left(\sin^2\theta\right)\left(\frac{\cos^2\theta}{\cos^2\theta}\right)$ Rewrite 1 with com denom

$= \frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta}$ Multiply

$= \frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta}$ Write as a single fraction

$= \frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}$ Factor the numerator

$= \frac{\sin^2\theta\sin^2\theta}{\cos^2\theta}$ Pythagorean Identity

$= \sin^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right)$ Factor

$= \sin^2\theta\tan^2\theta$ Quotient Identity

22. $\frac{1-\tan^2\theta}{1-\cot^2\theta} = \frac{\cos^2\theta - 1}{\cos^2\theta}$

SOLUTION:

$\frac{1-\tan^2\theta}{1-\cot^2\theta}$

$= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{\cos^2\theta}{\sin^2\theta}}$ Reciprocal Identities

$= \frac{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}}{\frac{\sin^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}$ Rewrite 1 with com denom

$= \frac{\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}{\frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta}}$ Write numerator and denominator with a common denominator

$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} \cdot \frac{\sin^2\theta}{\sin^2\theta - \cos^2\theta}$ Multiply by the reciprocal

$= \frac{\cos^2\theta - (1-\cos^2\theta)}{\cos^2\theta} \cdot \frac{1-\cos^2\theta}{(1-\cos^2\theta)-\cos^2\theta}$ Pythagorean Identity

$= \frac{-1+2\cos^2\theta}{\cos^2\theta} \cdot \frac{1-\cos^2\theta}{1-2\cos^2\theta}$ Simplify the numerator

$= \frac{-(1-2\cos^2\theta)}{\cos^2\theta} \cdot \frac{1-\cos^2\theta}{1-2\cos^2\theta}$ Factor out −1

$= \frac{-(1-2\cos^2\theta)\left(1-\cos^2\theta\right)}{\cos^2\theta\left(1-2\cos^2\theta\right)}$ Multiply

$= \frac{-(1-\cos^2\theta)}{\cos^2\theta}$ Divide out $\left(1-2\cos^2\theta\right)$

$= \frac{\cos^2\theta - 1}{\cos^2\theta}$ Simplify the numerator

23. $\frac{1+\csc\theta}{\sec\theta}$ = cos θ + cot θ

SOLUTION:

$\frac{1+\csc\theta}{\sec\theta}$

$= \frac{1+\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}}$ Reciprocal Identity

$= \frac{\frac{\sin\theta}{\sin\theta}+\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}}$ Rewrite 1 with com denom

$= \frac{\frac{\sin\theta+1}{\sin\theta}}{\frac{1}{\cos\theta}}$ Write the numerator as a single fraction

$= \frac{\sin\theta+1}{\sin\theta} \cdot \frac{\cos\theta}{1}$ Multiply by the reciprocal

$= \frac{\sin\theta\cos\theta + \cos\theta}{\sin\theta}$ Write as a single fraction

$= \frac{\sin\theta\cos\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$ Write as two fractions

$= \cos\theta + \frac{\cos\theta}{\sin\theta}$ Divide out $\sin\theta$

$= \cos\theta + \cot\theta$ Quotient Identity

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24. $(\csc\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

SOLUTION:

$(\csc\theta - \cot\theta)^2$	
$= (\csc\theta - \cot\theta)(\csc\theta - \cot\theta)$	Rewrite as a product
$= \csc^2\theta - 2\csc\theta\cot\theta + \cot^2\theta$	Multiply binomials.
$= \frac{1}{\sin^2\theta} - \frac{2}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin^2\theta}$	Reciprocal/Quotient Identities
$= \frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$	Multiply fractions
$= \frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}$	Write as a single fraction
$= \frac{1-2\cos\theta+\cos^2\theta}{1-\cos^2\theta}$	Pythagorean Identity
$= \frac{(1-\cos\theta)^2}{(1+\cos\theta)(1-\cos\theta)}$	Factor
$= \frac{1-\cos\theta}{1+\cos\theta}$	Divide out $(1-\cos\theta)$

25. $\dfrac{1+\tan^2\theta}{1-\tan^2\theta} = \dfrac{1}{2\cos^2\theta - 1}$

SOLUTION:

$\frac{1+\tan^2\theta}{1-\tan^2\theta}$	
$= \frac{1+\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}$	Quotient Identity
$= \frac{\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}}$	Rewrite 1 with com denom
$= \frac{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$	Write numerator and denominator with a common denominator
$= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}$	Multiply by the reciprocal
$= \frac{\cos^2\theta(\cos^2\theta+\sin^2\theta)}{\cos^2\theta(\cos^2\theta-\sin^2\theta)}$	Multiply.
$= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}$	Divide out $\cos^2\theta$.
$= \frac{1}{\cos^2\theta-\sin^2\theta}$	Pythagorean Identity
$= \frac{1}{\cos^2\theta-(1-\cos^2\theta)}$	Pythagorean Identity
$= \frac{1}{\cos^2\theta-1+\cos^2\theta}$	Distributive Property
$= \frac{1}{2\cos^2\theta-1}$	Simplify the denominator.

26. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$

SOLUTION:

$\tan^2\theta\cos^2\theta$	
$= \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta$	Quotient Identity
$= \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta}$	Multiply.
$= \sin^2\theta$	Divide out common factor of $\cos^2\theta$.
$= 1-\cos^2\theta$	Pythagorean Identity

27. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

SOLUTION:

$\sec\theta - \cos\theta$	
$= \frac{1}{\cos\theta} - \cos\theta$	Reciprocal Identity
$= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$	Rewrite $\cos\theta$ using the common denominator.
$= \frac{1-\cos^2\theta}{\cos\theta}$	Add fractions.
$= \frac{\sin^2\theta}{\cos\theta}$	Pythagorean Identity
$= \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{1}$	Rewrite as a procuct of two fractions.
$= \tan\theta\sin\theta$	Quotient Identity

28. $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta$

SOLUTION:

$1-\tan^4\theta$	
$= (1-\tan^2\theta)(1+\tan^2\theta)$	Factor difference of two squares.
$= [1-(\sec^2\theta-1)](\sec^2\theta)$	Pythagorean Identities
$= [1-\sec^2\theta+1)](\sec^2\theta)$	Distributive Property
$= (2-\sec^2\theta)(\sec^2\theta)$	Simplify.
$= 2\sec^2\theta-\sec^4\theta$	Distributive Property

29. $(\csc\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

SOLUTION:

$(\csc\theta-\cot\theta)^2$	
$= \left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)^2$	Reciprocal and Quotient Identities
$= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$	Add fractions.
$= \frac{(1-\cos\theta)^2}{\sin^2\theta}$	Power of a Quotient
$= \frac{(1-\cos\theta)^2}{1-\cos^2\theta}$	Pythagorean Identity
$= \frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$	Factor.
$= \frac{1-\cos\theta}{1+\cos\theta}$	Divide out common factor.

30. $\dfrac{1+\tan\theta}{\sin\theta+\cos\theta} = \sec\theta$

SOLUTION:

$\frac{1+\tan\theta}{\sin\theta+\cos\theta}$	
$= \frac{1+\frac{\sin\theta}{\cos\theta}}{\sin\theta+\cos\theta}$	Quotient Identity
$= \frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\sin\theta+\cos\theta}$	Rewrite 1 using the com denom
$= \frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\sin\theta+\cos\theta}$	Write the numerator as a single fraction
$= \frac{\cos\theta+\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta+\cos\theta}$	Multiply by the reciprocal
$= \frac{\cos\theta+\sin\theta}{\cos\theta(\sin\theta+\cos\theta)}$	Multiply
$= \frac{1}{\cos\theta}$	Divide out $(\sin\theta+\cos\theta)$
$= \sec\theta$	Reciprocal Identity

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31. $\dfrac{2+\csc\theta\sec\theta}{\csc\theta\sec\theta} = (\sin\theta + \cos\theta)^2$

SOLUTION:

$\dfrac{2+\csc\theta\sec\theta}{\csc\theta\sec\theta}$	
$= \dfrac{2}{\csc\theta\sec\theta} + \dfrac{\csc\theta\sec\theta}{\csc\theta\sec\theta}$	Write as a sum of two fractions.
$= \dfrac{2}{\csc\theta\sec\theta} + 1$	$\dfrac{\csc\theta\sec\theta}{\csc\theta\sec\theta} = 1$
$= 2\cdot\dfrac{1}{\csc\theta}\cdot\dfrac{1}{\sec\theta} + 1$	Write $\dfrac{2}{\csc\theta\sec\theta}$ as a product.
$= 2\sin\theta\cos\theta + (\sin^2\theta + \cos^2\theta)$	Reciprocal and Pythagorean Identities
$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$	Commutative Property of Addition
$= (\sin\theta + \cos\theta)^2$	Factor Perfect Square Trinomial.

32. **OPTICS** If two prisms of the same power are placed next to each other, their total power can be determined using $z = 2p\cos\theta$, where z is the combined power of the prisms, p is the power of the individual prisms, and θ is the angle between the two prisms. Verify that $2p\cos\theta = 2p(1 - \sin^2\theta)\sec\theta$.

SOLUTION:

$2p(1-\sin^2\theta)\sec\theta$	
$= 2p\cos^2\theta\sec\theta$	Pythagorean Identity
$= 2p\cos^2\theta\cdot\dfrac{1}{\cos\theta}$	Reciprocal Identity
$= 2p\cos\theta$	Divide out $\cos\theta$

33. **PHOTOGRAPHY** The amount of light passing through a polarization filter can be modeled using $I = I_m\cos^2\theta$, where I is the amount of light passing through the filter, I_m is the amount of light shined on the filter, and θ is the angle of rotation between the light source and the filter. Verify that $I_m\cos^2\theta = I_m - \dfrac{I_m}{\cot^2\theta + 1}$.

SOLUTION:

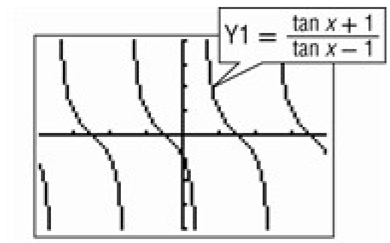
$I_m - \dfrac{I_m}{\cot^2\theta+1} = I_m\left(1 - \dfrac{1}{\cot^2\theta+1}\right)$	Factor.
$= I_m\left(1 - \dfrac{1}{\csc^2\theta}\right)$	Pythagorean Identity
$= I_m(1-\sin^2\theta)$	Reciprocal Identity
$= I_m\cos^2\theta$	Pythagorean Identity

GRAPHING CALCULATOR Test whether each equation is an identity by graphing. If it appears to be an identity, verify it. If not, find an *x*-value for which both sides are defined but not equal.

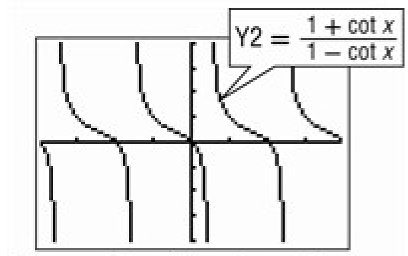
34. $\dfrac{\tan x + 1}{\tan x - 1} = \dfrac{1 + \cot x}{1 - \cot x}$

SOLUTION:

Graph $\mathbf{Y1} = \dfrac{\tan x + 1}{\tan x - 1}$ and then graph $\mathbf{Y2} = \dfrac{1+\cot x}{1-\cot x}$.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

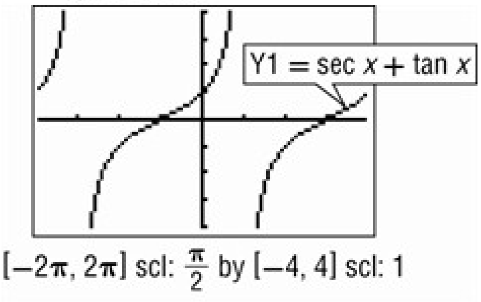
The graphs of the related functions do not coincide for all values of x for which both functions are defined.
Using the **intersect** feature from the **CALC** menu on the graphing calculator to find that when $x = \pi$, **Y1** = –1 and **Y2** is undefined. Therefore, the equation is not an identity.

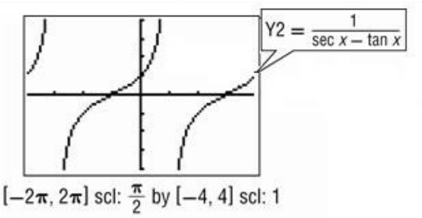
35. $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$

SOLUTION:

Graph $Y1 = \sec x + \tan x$ and then graph

$Y2 = \dfrac{1}{\sec x - \tan x}$.





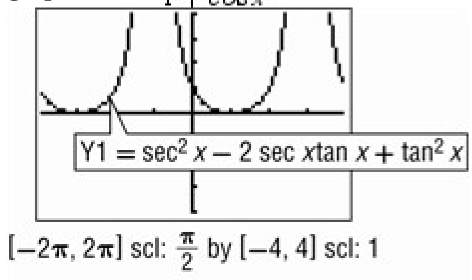
The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

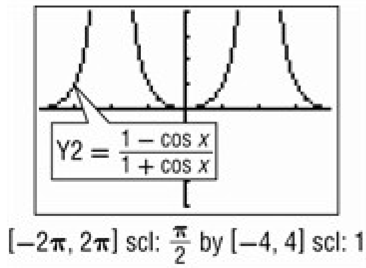
$\dfrac{1}{\sec x - \tan x} = \dfrac{1}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}}$	Reciprocal and Quotient Identities
$= \dfrac{1}{\frac{1 - \sin x}{\cos x}}$	Subtract fractions in the denominator.
$= \dfrac{\cos x}{1 - \sin x}$	$1 \div \dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$
$= \dfrac{\cos x}{1 - \sin x} \cdot \dfrac{1 + \sin x}{1 + \sin x}$	Multiply numerator and denominator by the conjugate of the denominator.
$= \dfrac{\cos x + \sin x \cos x}{1 - \sin^2 x}$	Multiply.
$= \dfrac{\cos x + \sin x \cos x}{\cos^2 x}$	Pythagorean Identity
$= \dfrac{\cos x}{\cos^2 x} + \dfrac{\sin x \cos x}{\cos^2 x}$	Write as a sum of two fractions.
$= \dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}$	Divide out the common factor $\cos x$.
$= \sec x + \tan x$	Reciprocal and Quotient Identities.

36. $\sec^2 x - 2 \sec x \tan x + \tan^2 x = \dfrac{1 - \cos x}{1 + \cos x}$

SOLUTION:

Graph **Y1** = $\sec^2 x - 2 \sec x \tan x + \tan^2 x$ and then graph **Y2** = $\dfrac{1 - \cos x}{1 + \cos x}$.





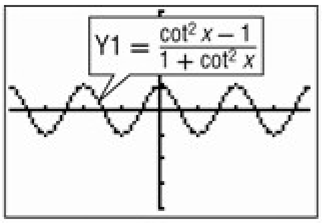
The graphs of the related functions do not coincide for all values of x for which both functions are defined.

Using the **intersect** feature from the **CALC** menu on the graphing calculator to find that when $x = 0$, **Y1** = 1 and **Y2** = 0. Therefore, the equation is not an identity.

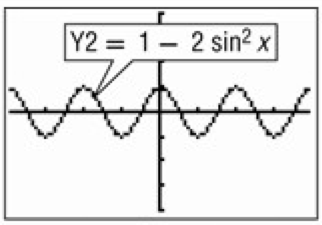
37. $\dfrac{\cot^2 x - 1}{1 + \cot^2 x} = 1 - 2\sin^2 x$

SOLUTION:

Graph **Y1** = $\dfrac{\cot^2 x - 1}{1 + \cot^2 x}$ and then graph **Y2** = $1 - 2\sin^2 x$.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

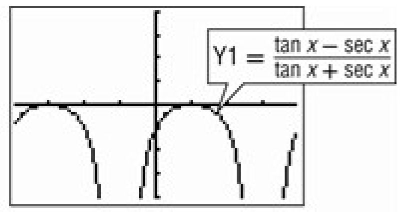
The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

$\dfrac{\cot^2 x - 1}{1 + \cot^2 x} = \dfrac{\cot^2 x - 1}{\csc^2 x}$	Pythagorean Identity
$= \dfrac{\dfrac{\cos^2 x}{\sin^2 x} - 1}{\dfrac{1}{\sin^2 x}}$	Quotient and Reciprocal Identities
$= \left(\dfrac{\cos^2 x}{\sin^2 x} - 1\right)\sin^2 x$	$1 \div \dfrac{1}{\sin^2 x} = \sin^2 x$
$= \cos^2 x - \sin^2 x$	Multiply and divide out common factor.
$= (1 - \sin^2 x) - \sin^2 x$	Pythagorean Identity
$= 1 - 2\sin^2 x$ ✓	Simplify.

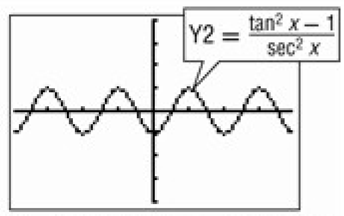
38. $\dfrac{\tan x - \sec x}{\tan x + \sec x} = \dfrac{\tan^2 x - 1}{\sec^2 x}$

SOLUTION:

Graph **Y1** = $\dfrac{\tan x - \sec x}{\tan x + \sec x}$ and then graph **Y2** = $\dfrac{\tan^2 x - 1}{\sec^2 x}$.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

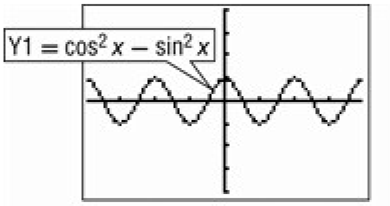
The graphs of the related functions do not coincide for all values of x for which both functions are defined.
Using the **intersect** feature from the **CALC** menu on the graphing calculator to find that when $x = \frac{\pi}{4}$, **Y1** ≈ -0.17 and **Y2** $= 0$. Therefore, the equation is not an identity.

5-2 Verifying Trigonometric Identities

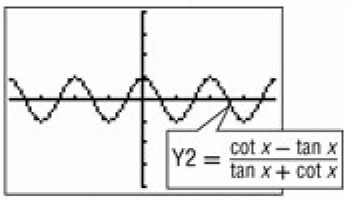
39. $\cos^2 x - \sin^2 x = \dfrac{\cot x - \tan x}{\tan x + \cot x}$

SOLUTION:

Graph **Y1** = $\cos^2 x - \sin^2 x$ and then graph **Y2** = $\dfrac{\cot x - \tan x}{\tan x + \cot x}$.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

$$
\begin{aligned}
&\frac{\cot x - \tan x}{\tan x + \cot x}\\
&= \frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} && \text{Quotient Identities}\\
&= \frac{\frac{\cos^2 x}{\sin x\cos x} - \frac{\sin^2 x}{\sin x\cos x}}{\frac{\sin^2 x}{\sin x\cos x} + \frac{\cos^2 x}{\sin x\cos x}} && \text{Multiply to get common denominators}\\
&= \frac{\frac{\cos^2 x - \sin^2 x}{\sin x\cos x}}{\frac{\sin^2 x + \cos^2 x}{\sin x\cos x}} && \text{Add fractions}\\
&= \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x} && \text{Multiply numerator and denominator by } (\sin x \cdot \cos x)\\
&= \cos^2 x - \sin^2 x && \text{Pythagorean Identity}
\end{aligned}
$$

Verify each identity.

40. $\sqrt{\dfrac{\sin x \tan x}{\sec x}} = |\sin x|$

SOLUTION:

$$
\begin{aligned}
&\sqrt{\frac{\sin x\tan x}{\sec x}}\\
&= \sqrt{\frac{\sin x\left(\frac{\sin x}{\cos x}\right)}{\frac{1}{\cos x}}} && \text{Quotient Identities}\\
&= \sqrt{\frac{\frac{\sin^2 x}{\cos x}}{\frac{1}{\cos x}}} && \text{Multiply.}\\
&= \sqrt{\frac{\sin^2 x}{\cos x}\cdot\frac{\cos x}{1}} && \text{Multiply by reciprocal}\\
&= \sqrt{\sin^2 x} && \text{Divide out } \cos x\\
&= |\sin x| && \text{Simplify the square root}
\end{aligned}
$$

41. $\sqrt{\dfrac{\sec x - 1}{\sec x + 1}} = \left|\dfrac{\sec x - 1}{\tan x}\right|$

SOLUTION:

$$
\begin{aligned}
&\sqrt{\frac{\sec x - 1}{\sec x + 1}}\\
&= \sqrt{\frac{\sec x - 1}{\sec x + 1}}\cdot\sqrt{\frac{\sec x - 1}{\sec x - 1}} && \text{Multiply by conjugate of the denominator}\\
&= \sqrt{\frac{(\sec x - 1)^2}{\sec^2 x - 1}} && \text{Multiply}\\
&= \sqrt{\frac{(\sec x - 1)^2}{\tan^2 x}} && \text{Pythagorean Identity}\\
&= \left|\frac{\sec x - 1}{\tan x}\right| && \text{Simplify the square root}
\end{aligned}
$$

42. $\ln|\csc x + \cot x| + \ln|\csc x - \cot x| = 0$

SOLUTION:

$$
\begin{aligned}
&\ln|\csc x + \cot x| + \ln|\csc x - \cot x|\\
&= \ln|(\csc x + \cot x)(\csc x - \cot x)| && \text{Product Property of Logarithms}\\
&= \ln\left|\csc^2 x - \cot^2 x\right| && \text{Multiply}\\
&= \ln|1| && \text{Pythagorean Identity}\\
&= 0 && \text{Simplify}
\end{aligned}
$$

43. $\ln|\cot x| + \ln|\tan x \cos x| = \ln|\cos x|$

SOLUTION:

$$
\begin{aligned}
&\ln|\cot x| + \ln|\tan x\cos x|\\
&= \ln\left|\frac{\cos x}{\sin x}\right| + \ln\left|\frac{\sin x}{\cos x}\cdot\cos x\right| && \text{Quotient Identities}\\
&= \ln\left|\frac{\cos x}{\sin x}\right| + \ln|\sin x| && \text{Divide out common factor } \cos x.\\
&= \ln\left|\frac{\cos x}{\sin x}\cdot\sin x\right| && \text{Product Property of Logarithms}\\
&= \ln|\cos x| \ \checkmark && \text{Divide out common factor } \sin x.
\end{aligned}
$$

5-2 Verifying Trigonometric Identities

Verify each identity.

44. $\sec^2\theta + \tan^2\theta = \sec^4\theta - \tan^4\theta$

SOLUTION:

Start with the right side of the identity.

$\sec^4\theta - \tan^4\theta$	
$= (\sec^2\theta + \tan^2\theta)(\sec^2\theta - \tan^2\theta)$	Factor.
$= (\sec^2\theta + \tan^2\theta)(\tan^2\theta + 1 - \tan^2\theta)$	Pythagorean Identity
$= (\sec^2\theta + \tan^2\theta)(1)$	Combine like terms.
$= \sec^2\theta + \tan^2\theta$ ✓	Multiplicative Identity

45. $-2\cos^2\theta = \sin^4\theta - \cos^4\theta - 1$

SOLUTION:

Start with the right side of the identity.

$\sin^4\theta - \cos^4\theta - 1$	
$= (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) - 1$	Factor $(\sin^4\theta - \cos^4\theta)$.
$= (1)(1 - \cos^2\theta - \cos^2\theta) - 1$	Pythagorean Identities
$= 1 - \cos^2\theta - \cos^2\theta - 1$	Multiply.
$= -2\cos^2\theta$ ✓	Add.

46. $\sec^2\theta\sin^2\theta = \sec^4\theta - (\tan^4\theta + \sec^2\theta)$

SOLUTION:

Start with the right side of the identity.

$\sec^4\theta - (\tan^4\theta + \sec^2\theta)$	
$= (\sec^4\theta - \tan^4\theta) - \sec^2\theta$	Distributive and Associative Properties
$= (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta) - \sec^2\theta$	Factor.
$= (1)(\sec^2\theta + \tan^2\theta) - \sec^2\theta$	Pythagorean Identity
$= \sec^2\theta + \tan^2\theta - \sec^2\theta$	Multiply.
$= \tan^2\theta$	Add.
$= \frac{\sin^2\theta}{\cos^2\theta}$	Quotient Identity
$= \frac{1}{\cos^2\theta} \cdot \frac{\sin^2\theta}{1}$	Write as a product.
$= \sec^2\theta\sin^2\theta$	Reciprocal Identity

47. $3\sec^2\theta\tan^2\theta + 1 = \sec^6\theta - \tan^6\theta$

SOLUTION:

Start with the right side of the identity.

$$\sec^6\theta - \tan^6\theta$$
$$= (\sec^3\theta - \tan^3\theta)(\sec^3\theta + \tan^3\theta)$$
$$= (\sec\theta - \tan\theta)(\sec^2\theta + \sec\theta\tan\theta + \tan^2\theta)(\sec\theta + \tan\theta)(\sec^2\theta - \sec\theta\tan\theta + \tan^2\theta)$$
$$= (\sec^2\theta - \tan^2\theta)[(1 + \tan^2\theta) + \sec\theta\tan\theta + \tan^2\theta]\cdot[(1 + \tan^2\theta) - \sec\theta\tan\theta + \tan^2\theta]$$
$$= (1 + 2\tan^2\theta + \sec\theta\tan\theta)(1 + 2\tan^2\theta - \sec\theta\tan\theta)$$
$$= (1 + 2\tan^2\theta)^2 - (\sec\theta\tan\theta)^2$$
$$= 1 + 4\tan^2\theta + 4\tan^4\theta - \sec^2\theta\tan^2\theta$$
$$= 1 + \tan^2\theta(4 + 4\tan^2\theta - \sec^2\theta)$$
$$= 1 + \tan^2\theta[4 + 4(\sec^2\theta - 1) - \sec^2\theta]$$
$$= 1 + \tan^2\theta(4 + 4\sec^2\theta - 4 - \sec^2\theta)$$
$$= 1 + \tan^2\theta(3\sec^2\theta)$$
$$= 1 + 3\tan^2\theta\sec^2\theta$$
$$= 3\sec^2\theta\tan^2\theta + 1$$

48. $\sec^4 x = 1 + 2\tan^2 x + \tan^4 x$

SOLUTION:

Start with the right side of the identity.

$1 + 2\tan^2 x + \tan^4 x$	
$= 1 + 2(\sec^2 x - 1) + (\sec^2 x - 1)^2$	Pythagorean Identities
$= 1 + 2\sec^2 x - 2 + \sec^4 x - 2\sec^2 x + 1$	Distribute and square.
$= \sec^4 x$ ✓	Combine like terms.

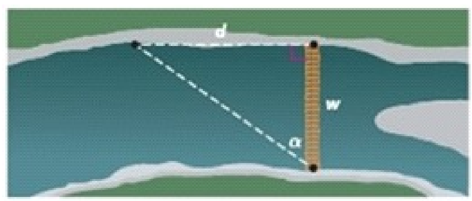
49. $\sec^2 x\csc^2 x = \sec^2 x + \csc^2 x$

SOLUTION:

Start with the left side of the identity.

$\sec^2 x\csc^2 x$	
$= (\tan^2 x + 1)\csc^2 x$	Pythagorean Identity
$= \left(\frac{\sin^2 x}{\cos^2 x} + 1\right)\left(\frac{1}{\sin^2 x}\right)$	Quotient and Reciprocal Identities
$= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$	Multiply.
$= \sec^2 x + \csc^2 x$ ✓	Reciprocal Identities

50. **ENVIRONMENT** A biologist studying pollution situates a net across a river and positions instruments at two different stations on the river bank to collect samples. In the diagram shown, *d* is the distance between the stations and *w* is width of the river.



a. Determine an equation in terms of tangent α that can be used to find the distance between the stations.

b. Verify that $d = \frac{w\cos(90°\ \alpha)}{\cos\alpha}$.

c. Complete the table shown for $d = 40$ feet.

w	20	40	60	80	100	120
α						

d. If $\alpha > 60°$ or $\alpha < 20°$, the instruments will not function properly. Use the table from part **c** to determine whether sites in which the width of the river is 5, 35, or 140 feet could be used for the experiment.

SOLUTION:

a.

$\tan\alpha = \frac{\text{opp}}{\text{adj}}$	Tangent ratio
$\tan\alpha = \frac{d}{w}$	opp = d and adj = w
$w\tan\alpha = d$	Multiply each side by w.
$d = w\tan\alpha$	Symmetric Property of Equality

b.

$d = w\tan\alpha$

$= \frac{w\sin\alpha}{\cos\alpha}$	Quotient Identity
$= \frac{w\cos(90° - \alpha)}{\cos\alpha}$	Cofunction Identity

c.

$\tan\alpha = \frac{d}{w}$

$\alpha = \tan^{-1}\left(\frac{d}{w}\right)$

$\alpha = \tan^{-1}\left(\frac{40}{20}\right) \approx 63.4$ $\quad$ $\alpha = \tan^{-1}\left(\frac{40}{80}\right) \approx 26.6$

$\alpha = \tan^{-1}\left(\frac{40}{40}\right) = 45$ $\quad$ $\alpha = \tan^{-1}\left(\frac{40}{100}\right) \approx 21.8$

$\alpha = \tan^{-1}\left(\frac{40}{60}\right) \approx 33.7$ $\quad$ $\alpha = \tan^{-1}\left(\frac{40}{120}\right) \approx 18.4$

w	20	40	60	80	100	120
α	63.4	45	33.7	26.6	21.8	18.4

d. If $w = 5$ then α will be greater than 63.4° since 5 < 20. If $w = 140$, then α will be less than 18.4° since 140 > 120. If $w = 35$, then 45° < α < 63.4° since 35 is between 20 and 40. The sites with widths of 5 and 140 feet could not be used because α > 60° and α < 20°, respectively. The site with a width of 35 feet could be used because 20° < α < 60°.

HYPERBOLIC FUNCTIONS The *hyperbolic trigonometric functions* are defined in the following ways.

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\operatorname{csch} x = \frac{1}{\sinh x}, \ x \neq 0$

$\operatorname{sech} x = \frac{1}{\cosh x}$

$\coth x = \frac{1}{\tanh x}, x \neq 0$

Verify each identity using the functions shown above.

51. $\cosh^2 x - \sinh^2 x = 1$

SOLUTION:

$\cosh^2 - \sinh^2 x$	Start with the left side.
$= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$	Replace cosh and sinh with definitions.
$= \frac{1}{4}[e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})]$	Factor and square each expression.
$= \frac{1}{4}[e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}]$	Distribute the negative.
$= \frac{1}{4}(4)$	Combine like terms.
$= 1$ ✓	Multiply.

52. $\sinh(-x) = -\sinh x$

SOLUTION:

$\sinh(-x)$	Start with the left side.
$= \frac{1}{2}[e^{-x} - e^{-(-x)}]$	Substitute $-x$ for x in defintion for sinh.
$= \frac{1}{2}(e^{-x} - e^{x})$	Simplify.
$= \frac{1}{2}(-e^{x} + e^{-x})$	Commutative Property of Addition
$= -\frac{1}{2}(e^{x} - e^{-x})$	Factor out -1.
$= -\left[\frac{1}{2}(e^{x} - e^{-x})\right]$	Associative Property of Multiplication
$= -\sinh x$ ✓	Substitute.

5-2 Verifying Trigonometric Identities

53. $\operatorname{sech}^2 x = 1 - \tanh^2 x$

SOLUTION:

$1-\tanh^2 x = 1-\dfrac{\sinh^2 x}{\cosh^2 x}$	Start with the right side.
$= \dfrac{\cosh^2 x}{\cosh^2 x} - \dfrac{\sinh^2 x}{\cosh^2 x}$	Replace tanh with $\frac{\sinh}{\cosh}$.
$= \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$	Change to common denominators.
$= \dfrac{\frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x-e^{-x})^2}{\cosh^2 x}$	Replace cosh and sinh with definitions.
$= \dfrac{\frac{1}{4}[e^{2x}+2+e^{-2x}-(e^{2x}-2+e^{-2x})]}{\cosh^2 x}$	Factor and square each expression.
$= \dfrac{\frac{1}{4}(4)}{\cosh^2 x}$	Combine like terms.
$= \dfrac{1}{\cosh^2 x}$	Multiply.
$= \operatorname{sech}^2 x$	Replace $\frac{1}{\cosh x}$ with sech$x$.

54. $\cosh(-x) = \cosh x$

SOLUTION:

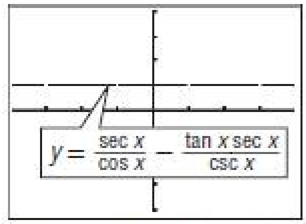
$\cosh(-x)$	Start with the left side.
$=\frac{1}{2}[e^{-x}+e^{-(-x)}]$	Substitute $-x$ for x in definition for cosh.
$=\frac{1}{2}(e^{-x}+e^{x})$	Simplify.
$=\frac{1}{2}(e^{x}+e^{-x})$	Commutative Property of Addition
$=\cosh x$ ✓	Substitute.

GRAPHING CALCULATOR Graph each side of each equation. If the equation appears to be an identity, verify it algebraically.

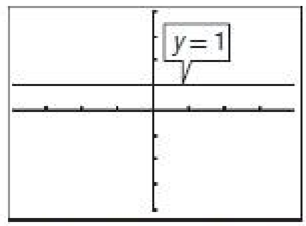
55. $\dfrac{\sec x}{\cos x} - \dfrac{\tan x \sec x}{\csc x} = 1$

SOLUTION:

Graph **Y1** = $\dfrac{\sec x}{\cos x} - \dfrac{\tan x \sec x}{\csc x}$ and **Y2** = 1.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

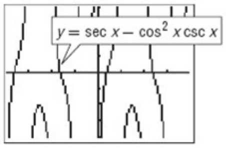
The graphs appear to be the same, so the equation appears to be an identity. Verify this algebraically.

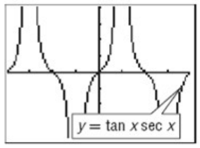
$\dfrac{\sec x}{\cos x} - \dfrac{\tan x \sec x}{\csc x}$	Start with the left side of the identity.
$= \dfrac{1}{\cos x}\cdot\dfrac{1}{\cos x} - \dfrac{\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}}{\frac{1}{\sin x}}$	Quotient and Reciprocal Identities
$= \dfrac{1}{\cos^2 x} - \dfrac{\frac{\sin x}{\cos^2 x}}{\frac{1}{\sin x}}$	Multiply fractions.
$= \dfrac{1}{\cos^2 x} - \dfrac{\sin^2 x}{\cos^2 x}$	Multiply by reciprocal of the denominator.
$= \sec^2 x - \tan^2 x$	Reciprocal and Quotient Identity
$= 1$	Pythagorean Identity

5-2 Verifying Trigonometric Identities

56. sec *x* – cos^2 *x* csc *x* = tan *x* sec *x*

SOLUTION:



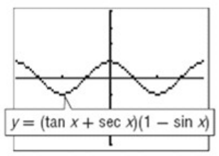


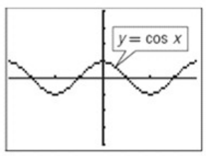
$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1 $[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

Graphs are not the same so sec *x* – cos^2 *x* csc *x* $\neq$ tan *x* sec*x*.

57. (tan *x* + sec *x*)(1 – sin *x*) = cos *x*

SOLUTION:



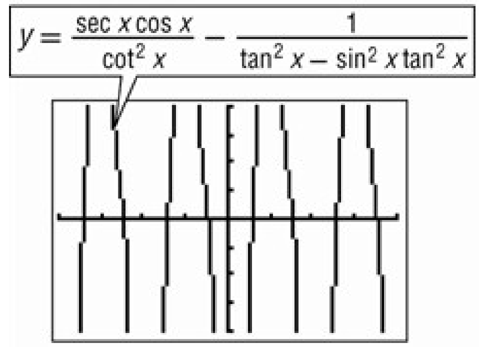


$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1 $[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

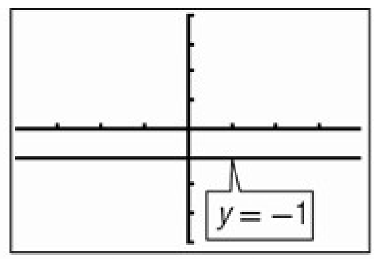
$(\tan x + \sec x)(1 - \sin x)$	Start with the left side.
$= \tan x - \tan x \sin x + \sec x - \sec x \sin x$	Multiply binomials.
$= \tan x - \frac{\sin x}{\cos x} \cdot \sin x + \frac{1}{\cos x} - \frac{1}{\cos x} \cdot \sin x$	Quotient and Reciprocal Identities
$= \tan x - \frac{\sin^2 x}{\cos x} + \frac{1}{\cos x} - \frac{\sin x}{\cos x}$	Multiply.
$= \tan x - \frac{\sin^2 x}{\cos x} + \frac{1}{\cos x} - \tan x$	Quotient Identity
$= -\frac{\sin^2 x}{\cos x} + \frac{1}{\cos x}$	$\tan x - \tan x = 0$
$= \frac{1}{\cos x} + \frac{-\sin^2 x}{\cos x}$	Commutative Property
$= \frac{1 - \sin^2 x}{\cos x}$	Add fractions.
$= \frac{\cos^2 x}{\cos x}$	Pythagorean Identity
$= \cos x$ ✓	Divide out common factor.

58. $\frac{\sec x \cos x}{\cot^2 x} - \frac{1}{\tan^2 x - \sin^2 x \tan^2 x} = -1$

SOLUTION:



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-4, 4]$ scl: 1

The graphs are not the same, so

$$\frac{\sec x \cos x}{\cot^2 x} - \frac{1}{\tan^2 x - \sin^2 x \tan^2 x} \neq -1$$

59. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate methods used to solve trigonometric equations. Consider 1 = 2 sin *x*.

a. NUMERICAL Isolate the trigonometric function in the equation so that sin *x* is the only expression on one side of the equation.

b. GRAPHICAL Graph the left and right sides of the equation you found in part **a** on the same graph over [0, 2π). Locate any points of intersection and express the values in terms of radians.

c. GEOMETRIC Use the unit circle to verify the answers you found in part **b**.

d. GRAPHICAL Graph the left and right sides of the equation you found in part **a** on the same graph over –2π < *x* < 2π. Locate any points of intersection and express the values in terms of radians.

e. VERBAL Make a conjecture as to the solutions of 1 = 2 sin *x*. Explain your reasoning.

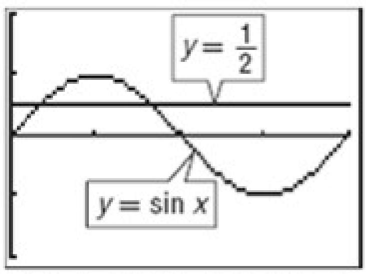
SOLUTION:

a. $2\sin x = 1$

$\frac{2\sin x}{2} = \frac{1}{2}$

$\sin x = \frac{1}{2}$

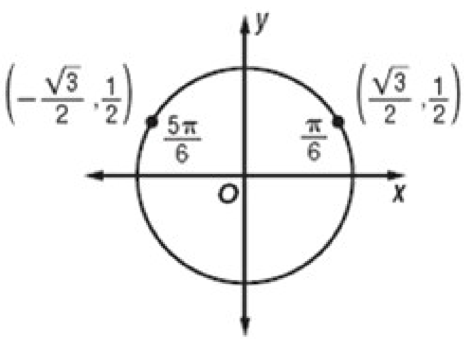
b.



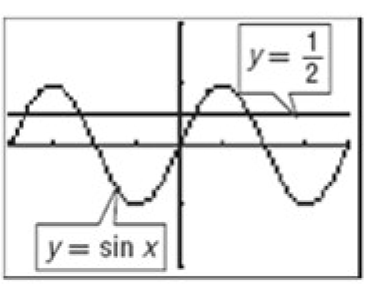
$[0, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 2]$ scl: 1

The graphs of $y = \sin x$ and $y = \frac{1}{2}$ intersect at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ over $[0, 2\pi)$.

c. On the unit circle, $\sin x$ refers to the y-coordinate. The y coordinate is $\frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



d.



$[-2\pi, 2\pi]$ scl: $\frac{\pi}{2}$ by $[-2, 2]$ scl: 1

The graphs of $y = \sin x$ and $y = \frac{1}{2}$ intersect at $-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}$ and $\frac{5\pi}{6}$ over $(-2\pi, 2\pi)$.

e. Since sine is a periodic function, with period 2π, the solutions repeat every $2n\pi$ from $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus, the solutions of $\sin x = \frac{1}{2}$ are $x = \frac{\pi}{6} + 2n\pi$ and $x = \frac{5\pi}{6} + 2n\pi$, where n is an integer.

60. **REASONING** Can substitution be used to determine whether an equation is an identity? Explain your reasoning.

SOLUTION:

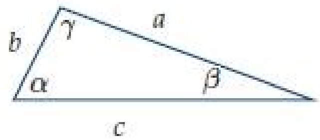
Substitution can be used to determine whether an equation is *not* an identity. However, this method cannot be used to determine whether an equation is an identity, because there is no way to prove that the identity is true for the entire domain.

61. **CHALLENGE** Verify that the area A of a triangle is given by

$$A = \frac{\alpha^2 \sin \beta \sin \gamma}{2\sin(\beta + \gamma)},$$

where a, b, and c represent the sides of the triangle and α, β, and γ are the respective opposite angles.

SOLUTION:



Use the Law of Sines to write an equation for b in terms of side a and angles B and α.

$\frac{\sin \beta}{b} = \frac{\sin \alpha}{a}$, so $b = \frac{a\sin \beta}{\sin \alpha}$.

$$A = \frac{1}{2}ab\sin \gamma$$
$$= \frac{1}{2}a\left(\frac{a\sin \beta}{\sin \alpha}\right)\sin \gamma$$
$$= \frac{a^2\sin \beta \sin \gamma}{2\sin \alpha}$$
$$= \frac{a^2\sin \beta \sin \gamma}{2\sin\,[180° - (\beta + \gamma)]}$$
$$= \frac{a^2\sin \beta \sin \gamma}{2[\sin 180° \cos(\beta + \gamma) - \cos 180° \sin(\beta + \gamma)]}$$
$$= \frac{a^2\sin \beta \sin \gamma}{2[0 \cdot \cos(\beta + \gamma) - (-1)\sin(\beta + \gamma)]}$$
$$= \frac{a^2\sin \beta \sin \gamma}{2\sin(\beta + \gamma)}$$

62. ***Writing in Math*** Use the properties of logarithms to explain why the sum of the natural logarithm of the six basic trigonometric functions for any angle θ is 0.

SOLUTION:

$$\ln|\sin x| + \ln|\cos x| + \ln|\tan x|$$
$$+ \ln|\csc x| + \ln|\sec x| + \ln|\cot x|$$
$$= \ln|(\sin x)(\csc x)(\cos x)(\sec x)(\tan x)(\cot x)|$$
$$= \ln|1|$$
$$= 0$$

According to the Product Property of Logarithms, the sum of the logarithms of the basic trigonometric functions is equal to the logarithm of the product. Since the product of the absolute values of the functions is 1, the sum of the logarithms is ln 1 or 0.

63. **OPEN ENDED** Create identities for sec x and csc x in terms of two or more of the other basic trigonometric functions.

SOLUTION:

Start with the identities sec x = sec x and csc x = csc x. Manipulate one side of each equation until you come up with an identity with two or more trigonometric functions.

Two possible identities are: tan x sin x + cos x = sec x and sin x + cot x cos x = csc x.

$$\begin{aligned}\tan x \sin x + \cos x &= \frac{\sin x}{\cos x}\cdot \sin x + \cos x\\ &= \frac{\sin^2 x}{\cos x} + \cos x\\ &= \frac{1-\cos^2 x}{\cos x} + \cos x\\ &= \frac{1}{\cos x} - \cos x + \cos x\\ &= \frac{1}{\cos x}\\ &= \sec x\end{aligned}$$

$$\begin{aligned}\sin x + \cot x \cos x &= \sin x + \frac{\cos x}{\sin x}\cdot \cos x\\ &= \sin x + \frac{\cos^2 x}{\sin x}\\ &= \sin x + \frac{1-\sin^2 x}{\sin x}\\ &= \sin x + \frac{1}{\sin x} - \sin x\\ &= \frac{1}{\sin x}\\ &= \csc x\end{aligned}$$

64. **REASONING** If two angles α and β are complementary, is $\cos^2 \alpha + \cos^2 \beta = 1$? Explain your reasoning. Justify your answers.

SOLUTION:

If α and β are complementary angles, then $\alpha + \beta = 90°$ and hence $\beta = 90° - \alpha$. Use this to make substitution in the equation.

$$\begin{aligned}&\cos^2 \alpha + \cos^2 \beta\\ &= \cos^2 \alpha + \cos^2(90° - \alpha)\\ &= \cos^2 \alpha + \sin^2 \alpha = 1.\end{aligned}$$

65. ***Writing in Math*** Explain how you would verify a trigonometric identity in which both sides of the equation are equally complex.

SOLUTION:

You can break the identity into two steps. For example consider the identity:

$$\frac{1-\tan^2 x}{1-\cot^2 x} = \frac{\cos^2 - 1}{\cos^2 x}$$

First manipulate one side of the equation until it is simplified reasonably.

$$\begin{aligned}\frac{1-\tan^2 x}{1-\cot^2 x} &= \frac{\cos^2 - 1}{\cos^2 x}\\ &= \frac{-\left(1-\cos^2 x\right)}{\cos^2 x}\\ &= \frac{-\sin^2 x}{\cos^2 x}\\ &= -\tan^2 x\end{aligned}$$

Since all steps performed were legitimate operations, we know that the following identity holds:

$$-\tan^2 x = \frac{1-\tan^2 x}{1-\cot^2 x}$$

Now perform operations on the left side to verify.

$$
\begin{aligned}
-\tan^2 x &= \frac{1-\tan^2 x}{1-\cot^2 x} \\
&= \frac{-\tan^2 x\left(-\frac{1}{\tan^2 x}+1\right)}{1-\cot^2 x} \\
&= \frac{-\tan^2 x\ \cancel{(1-\cot^2 x)}}{\cancel{1-\cot^2 x}} \\
&= -\tan^2 x
\end{aligned}
$$

We have shown $\frac{\cos^2 x-1}{\cos^2 x} = -\tan^2 x$ and $\frac{1-\tan^2 x}{1-\cot^2 x} = -\tan^2 x$ by the transitive property of equality, we have verified that $\frac{1-\tan^2 x}{1-\cot^2 x} = \frac{\cos^2 -1}{\cos^2 x}$.

Simplify each expression.

66. $\cos\ \theta\ \csc\ \theta$

SOLUTION:

$\cos\theta\csc\theta = \cos\theta\cdot\frac{1}{\sin\theta}$	Reciprocal Identity
$= \frac{\cos\theta}{\sin\theta}$	Multiply.
$= \cot\theta$	Quotient Identity

67. $\tan\ \theta\ \cot\ \theta$

SOLUTION:

$\tan\theta\cot\theta = \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$	Quotient Identities
$= 1$	Divide out the common factors.

68. $\sin\ \theta \cot\ \theta$

SOLUTION:

$\sin\theta\cot\theta = \sin\theta\cdot\frac{\cos\theta}{\sin\theta}$	Quotient Identity
$= \cos\theta$	Divide out the common factors.

69. $\frac{\cos\ \theta \csc\ \theta}{\tan\ \theta}$

SOLUTION:

$\frac{\cos\theta\csc\theta}{\tan\theta} = \frac{\cos\theta\cdot\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}}$	Reciprocal and Quotient Identities
$= \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$	Multiply by reciprocal of the denominator.
$= \frac{\cos^2\theta}{\sin^2\theta}$	Multiply fractions.
$= \cot^2\theta$	Quotient Identity

70. $\frac{\sin\ \theta \csc\ \theta}{\cot\ \theta}$

SOLUTION:

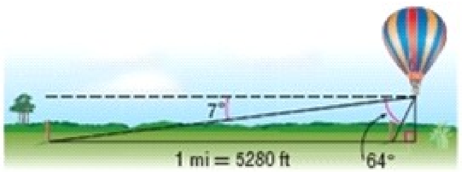
$\frac{\sin\theta\csc\theta}{\cot\theta} = \frac{\sin\theta\cdot\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$	Reciprocal and Quotient Identities
$= \frac{1}{1}\cdot\frac{\sin\theta}{\cos\theta}$	Divide out common factor of $\sin\theta$ and multiply by reciprocal of denominator.
$= \frac{\sin\theta}{\cos\theta}$	Multiply.
$= \tan\theta$	Quotient Identity

71. $\frac{1-\cos^2\theta}{\sin^2\theta}$

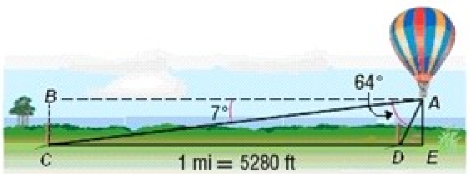
SOLUTION:

$\frac{1-\cos^2\theta}{\sin^2\theta} = \frac{(\sin^2\theta+\cos^2\theta)-\cos^2\theta}{\sin^2\theta}$	Pythagorean Identity
$= \frac{\sin^2\theta}{\sin^2\theta}$	Combine like terms.
$= 1$	Divide out the common factor of $\sin^2\theta$.

72. **BALLOONING** As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts, the angles of depression are 64° and 7°. How high is the balloon to the nearest foot?



SOLUTION:



First, find the measures of $\angle CAD$, $\angle BCA$, $\angle ACD$ and $\angle DAE$.

$$m\angle CAD = 64° - 7° = 57°$$

$$m\angle BCA = 180° - (7° + 90°) = 83°$$

$$m\angle ACD = 90° - 83° = 7°$$

$$m\angle DAE = 90° - 64° = 26°$$

In $\triangle ACD$, use the law of sines to find the length of $\overline{AD}$.

$$\frac{\sin 57°}{5280} = \frac{\sin 7°}{AD}$$

$$AD = \frac{5280 \sin 7°}{\sin 57°} \text{ or about } 767.3 \text{ feet}$$

Next, use right triangle ADE and the cosine function to find the length of $\overline{AE}$.

$$\cos 26° = \frac{AE}{AD}$$

$$\cos 26° = \frac{AE}{767.3}$$

$$AE = 767.3 \cos 26° \text{ or about } 690 \text{ ft}$$

Locate the vertical asymptotes, and sketch the graph of each function.

73. $y = \frac{1}{4} \tan x$

SOLUTION:

The graph of $y = \frac{1}{4} \tan x$ is the graph of $y = \tan x$ compressed vertically. The period is $\frac{\pi}{|1|}$ or π. Find the location of two consecutive vertical asymptotes.

$$bx + c = -\frac{\pi}{2} \qquad bx + c = \frac{\pi}{2}$$

$$(1)x + 0 = -\frac{\pi}{2} \quad \text{and} \quad (1)x + 0 = \frac{\pi}{2}$$

$$x = -\frac{\pi}{2} \qquad x = \frac{\pi}{2}$$

Create a table listing the coordinates of key points for $y = \frac{1}{4} \tan x$ for one period on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Function	$y = \tan x$	$y = \frac{1}{4} \tan x$
Vertical Asymptote	$x = -\frac{\pi}{2}$	$x = -\frac{\pi}{2}$
Intermediate Point	$\left(-\frac{\pi}{4}, -1\right)$	$\left(-\frac{\pi}{4}, -\frac{1}{4}\right)$
x-int	(0, 0)	(0, 0)
Intermediate Point	$\left(\frac{\pi}{4}, 1\right)$	$\left(\frac{\pi}{4}, \frac{1}{4}\right)$
Vertical Asymptote	$x = \frac{\pi}{2}$	$x = \frac{\pi}{2}$

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

74. $y = \csc 2x$

SOLUTION:

The graph of $y = \csc 2x$ is the graph of $y = \csc x$ compressed horizontally. The period is $\frac{2\pi}{|2|}$ or π.

Find the location of two vertical asymptotes.

$$\begin{aligned} bx + c &= -\pi \\ (2)x + 0 &= -\pi \\ x &= -\frac{\pi}{2} \end{aligned} \quad \text{and} \quad \begin{aligned} bx + c &= \pi \\ (2)x + 0 &= \pi \\ x &= \frac{\pi}{2} \end{aligned}$$

Create a table listing the coordinates of key points for $y = \csc 2x$ for one period on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Function	$y = \csc x$	$y = \csc 2x$
Vertical Asymptote	$x = -\pi$	$x = -\frac{\pi}{2}$
Intermediate Point	$\left(-\frac{\pi}{2}, -1\right)$	$\left(-\frac{\pi}{4}, -1\right)$
x-int	$x = 0$	$x = 0$
Intermediate Point	$\left(\frac{\pi}{2}, 1\right)$	$\left(\frac{\pi}{4}, 1\right)$
Vertical Asymptote	$x = \pi$	$x = \frac{\pi}{2}$

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

75. $y = \frac{1}{2} \sec 3x$

SOLUTION:

The graph of $y = \frac{1}{2}\sec 3x$ is the graph of $y = \sec x$ compressed vertically and horizontally. The period is $\frac{2\pi}{|3|}$ or $\frac{2\pi}{3}$. Find the location of two vertical asymptotes.

$$\begin{aligned} bx + c &= -\frac{\pi}{2} \\ 3x + 0 &= -\frac{\pi}{2} \\ 3x &= -\frac{\pi}{2} \\ x &= -\frac{\pi}{6} \end{aligned} \quad \text{and} \quad \begin{aligned} bx + c &= \frac{3\pi}{2} \\ 3x + 0 &= \frac{3\pi}{2} \\ 3x &= \frac{3\pi}{2} \\ x &= \frac{\pi}{2} \end{aligned}$$

Create a table listing the coordinates of key points for $y = \frac{1}{2}\sec 3x$ for one period on $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Function	$y = \sec x$	$y = \frac{1}{2}\sec 3x$
Vertical Asymptote	$x = -\frac{\pi}{2}$	$x = -\frac{\pi}{6}$
Intermediate Point	$(0, 1)$	$\left(0, \frac{1}{2}\right)$
x-int	$x = \frac{\pi}{2}$	$x = \frac{\pi}{3}$
Intermediate Point	$(\pi, -1)$	$\left(\frac{\pi}{3}, -\frac{1}{2}\right)$
Vertical Asymptote	$x = \frac{3\pi}{2}$	$x = \frac{\pi}{2}$

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

Write each degree measure in radians as a multiple of π and each radian measure in degrees.

76. $660°$

SOLUTION:

$$660° = 660°\left(\frac{\pi \text{radians}}{180°}\right)$$
$$= \frac{11\pi}{3} \text{radians}$$
$$= \frac{11\pi}{3}$$

77. $570°$

SOLUTION:

$$570° = 570°\left(\frac{\pi \text{radians}}{180^{\circ}}\right)$$
$$= \frac{19\pi}{6} \text{radians}$$
$$= \frac{19\pi}{6}$$

78. $158°$

SOLUTION:

$$158° = 158°\left(\frac{\pi \text{radians}}{180}\right)$$
$$= \frac{79\pi}{90} \text{radians}$$
$$= \frac{79\pi}{90}$$

79. $\frac{29\pi}{4}$

SOLUTION:

$$\frac{29\pi}{4} = \frac{29\pi}{4} \text{ radians}$$
$$= \frac{29\pi}{4} \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right)$$
$$= \frac{5220°}{4}$$
$$= 1305°$$

80. $\frac{17\pi}{6}$

SOLUTION:

$$\frac{17\pi}{6} = \frac{17\pi}{6} \text{ radians}$$
$$= \frac{17\pi}{6} \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right)$$
$$= 510°$$

81. 9

SOLUTION:

$$9 = 9 \text{ radians}$$
$$= 9 \text{ radians}\left(\frac{180°}{\pi \text{ radians}}\right)$$
$$= \frac{1620°}{\pi}$$
$$\approx 515.7°$$

Solve each inequality.

82. $x^2 - 3x - 18 > 0$

SOLUTION:

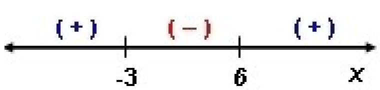
Let $f(x) = x^2 - 3x - 18$
$= (x + 3)(x - 6)$

$f(x)$ has real zeros at $x = -3$ and $x = 6$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

for $[-\infty, -3]$, $x = -4$
$(x + 3)(x - 6) \ ? \ 0$
$(-4 + 3)(-4 - 6) \ ? \ 0$
$(-1)(-10) \ ? \ 0$
$10 > 0$

for $[-3, 6]$, $x = 0$
$(x + 3)(x - 6) \ ? \ 0$
$(0 + 3)(0 - 6) \ ? \ 0$
$(3)(-6) \ ? \ 0$
$-18 < 0$

for $[6, \infty]$, $x = 8$
$(x + 3)(x - 6) \ ? \ 0$
$(8 + 3)(8 - 6) \ ? \ 0$
$(11)(2) \ ? \ 0$
$22 > 0$



The solutions of $x^2 - 3x - 18 > 0$ are x-values such that $f(x)$ is positive. From the sign chart, you can see that the solution set is $(-\infty, -3) \cup (6, \infty)$.

83. $x^2 + 3x - 28 < 0$

SOLUTION:

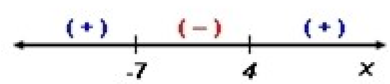
Let $f(x) = x^2 + 3x - 28$
$= (x + 7)(x - 4)$

$f(x)$ has real zeros at $x = -7$ and $x = 4$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

for $[-\infty, -7]$, $x = -8$
$(x + 7)(x - 4) \ ? \ 0$
$(-8 + 7)(-8 - 4) \ ? \ 0$
$(-1)(-12) \ ? \ 0$
$12 > 0$

for $[-7, 4]$, $x = 0$
$(x + 7)(x - 4) \ ? \ 0$
$(0 + 7)(0 - 4) \ ? \ 0$
$(7)(-4) \ ? \ 0$
$-28 < 0$

for $[4, \infty]$, $x = 8$
$(x + 7)(x - 4) \ ? \ 0$
$(8 + 7)(8 - 4) \ ? \ 0$
$(15)(4) \ ? \ 0$
$60 > 0$



The solutions of $x^2 + 3x - 28 < 0$ are x-values such that $f(x)$ is negative. From the sign chart, you can see that the solution set is $(-7, 4)$.

84. $x^2 - 4x \leq 5$

SOLUTION:

First, write $x^2 - 4x \leq 5$ as $x^2 - 4x - 5 \leq 0$.

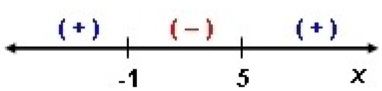
$$\text{Let } f(x) = x^2 - 4x - 5$$
$$= (x+1)(x-5)$$

$f(x)$ has real zeros at $x = -1$ and $x = 5$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$\text{for } [-\infty, -1], x = -2$$
$$(x+1)(x-5) \ ?\ 0$$
$$(-2+1)(-2-5) \ ?\ 0$$
$$(-1)(-7) \ ?\ 0$$
$$7 \geq 0$$

$$\text{for } [-1, 5], x = 0$$
$$(x+1)(x-5) \ ?\ 0$$
$$(0+1)(0-5) \ ?\ 0$$
$$(1)(-5) \ ?\ 0$$
$$-5 \leq 0$$

$$\text{for } [5, \infty], x = 6$$
$$(x+1)(x-5) \ ?\ 0$$
$$(6+1)(6-5) \ ?\ 0$$
$$(7)(1) \ ?\ 0$$
$$7 \geq 0$$



The solutions of $x^2 - 4x - 5 \leq 0$ are x-values such that $f(x)$ is negative or equal to 0. From the sign chart, you can see that the solution set is $[-1, 5]$.

85. $x^2 + 2x \geq 24$

SOLUTION:

First, write $x^2 + 2x \geq 24$ as $x^2 + 2x - 24 \geq 0$.

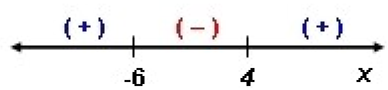
$$\text{Let } f(x) = x^2 + 2x - 24$$
$$= (x+6)(x-4)$$

$f(x)$ has real zeros at $x = -6$ and $x = 4$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$\text{for } [-\infty, -6], x = -8$$
$$(x+6)(x-4) \ ?\ 0$$
$$(-8+6)(-8-4) \ ?\ 0$$
$$(-2)(-12) \ ?\ 0$$
$$24 \geq 0$$

$$\text{for } [-6, 4], x = 0$$
$$(x+6)(x-4) \ ?\ 0$$
$$(0+6)(0-4) \ ?\ 0$$
$$(6)(-4) \ ?\ 0$$
$$-24 < 0$$

$$\text{for } [4, \infty], x = 8$$
$$(x+6)(x-4) \ ?\ 0$$
$$(8+6)(8-4) \ ?\ 0$$
$$(14)(2) \ ?\ 0$$
$$28 \geq 0$$



The solutions of $x^2 + 2x - 24 \geq 0$ are x-values such that $f(x)$ is positive or equal to 0. From the sign chart, you can see that the solution set is $(-\infty, -6] \cup [4, \infty)$.

86. $-x^2 - x + 12 \geq 0$

SOLUTION:

First, write $-x^2 - x + 12 \geq 0$ as $x^2 + x - 12 \leq 0$.

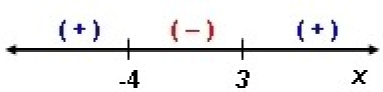
$$\begin{aligned} \text{Let } f(x) &= x^2 + x - 12 \\ &= (x+4)(x-3) \end{aligned}$$

$f(x)$ has real zeros at $x = -4$ and $x = 3$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$\begin{aligned} &\text{for } [-\infty, -4], x = -6 \\ &(x+4)(x-3) \ ?\ 0 \\ &(-6+4)(-6-3) \ ?\ 0 \\ &(-2)(-9) \ ?\ 0 \\ &18 > 0 \end{aligned}$$

$$\begin{aligned} &\text{for } [-4, 3], x = 0 \\ &(x+4)(x-3) \ ?\ 0 \\ &(0+4)(0-3) \ ?\ 0 \\ &(4)(-3) \ ?\ 0 \\ &-12 \leq 0 \end{aligned}$$

$$\begin{aligned} &\text{for } [3, \infty], x = 4 \\ &(x+4)(x-3) \ ?\ 0 \\ &(4+4)(4-3) \ ?\ 0 \\ &(8)(1) \ ?\ 0 \\ &8 > 0 \end{aligned}$$



The solutions of $x^2 + x - 12 \leq 0$ are x-values such that $f(x)$ is negative or equal to 0. From the sign chart, you can see that the solution set is $[-4, 3]$.

87. $-x^2 - 6x + 7 \leq 0$

SOLUTION:

First, write $-x^2 - 6x + 7 \leq 0$ as $x^2 + 6x - 7 \geq 0$.

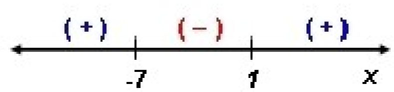
$$\begin{aligned} \text{Let } f(x) &= x^2 + 6x - 7 \\ &= (x+7)(x-1) \end{aligned}$$

$f(x)$ has real zeros at $x = -7$ and $x = 1$. Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if $f(x)$ is positive or negative at that point.

$$\begin{aligned} &\text{for } [-\infty, -7], x = -8 \\ &(x+7)(x-1) \ ?\ 0 \\ &(-8+7)(-8-1) \ ?\ 0 \\ &(-1)(-9) \ ?\ 0 \\ &9 \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{for } [-7, 1], x = 0 \\ &(x+7)(x-1) \ ?\ 0 \\ &(0+7)(0-1) \ ?\ 0 \\ &(7)(-1) \ ?\ 0 \\ &-7 < 0 \end{aligned}$$

$$\begin{aligned} &\text{for } [\infty, -7], x = -8 \\ &(x+7)(x-1) \ ?\ 0 \\ &(-8+7)(-8-1) \ ?\ 0 \\ &(-1)(-9) \ ?\ 0 \\ &9 \geq 0 \end{aligned}$$



The solutions of $x^2 + 6x - 7 \geq 0$ are x-values such that $f(x)$ is positive or equal to 0. From the sign chart, you can see that the solution set is $(-\infty, -7] \cup [1, \infty)$.

88. **FOOD** The manager of a bakery is randomly checking slices of cake prepared by employees to ensure that the correct amount of flavor is in each slice. Each 12-ounce slice should contain half chocolate and half vanilla flavored cream. The amount of chocolate by which each slice varies can be represented by $g(x) = \frac{1}{2}|x - 12|$. Describe the transformations in the function. Then graph the function.

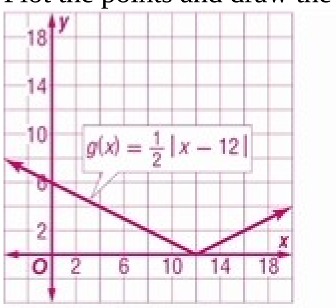
SOLUTION:

The parent function of $g(x)$ is $f(x) = |x|$. The factor of $\frac{1}{2}$ will cause the graph to be compressed since $\left|\frac{1}{2}\right| < 1$ and the subtraction of 12 will translate the graph 12 units to the right.

Make a table of values for x and $g(x)$.

x	4	8	12	16	20
$g(x)$	4	2	0	2	4

Plot the points and draw the graph of $g(x)$.



89. **SAT/ACT**

a, b, a, b, b, a, b, b, b, a, b, b, b, b, a, …

If the sequence continues in this manner, how many *b*s are there between the 44th and 47th appearances of the letter *a*?

A 91
B 135
C 138
D 182
E 230

SOLUTION:

The number of *b*s after each *a* is the same as the number of *a* in the list (i.e., after the 44th *a* there are 44 *b*s). Between the 44th and 47th appearances of *a* the number of *b*s will be 44 + 45 + 46 or 135. Therefore, the correct answer choice is B.

90. Which expression can be used to form an identity with $\frac{\sec\theta + \csc\theta}{1 + \tan\theta}$, when $\tan\theta \neq -1$?

F $\sin\theta$
G $\cos\theta$
H $\tan\theta$
J $\csc\theta$

SOLUTION:

$$\frac{\sec\theta + \csc\theta}{1 + \tan\theta} = \frac{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}{1 + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\sin\theta}{\cos\theta\sin\theta} + \frac{\cos\theta}{\sin\theta\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}}{\frac{\cos\theta + \sin\theta}{\cos\theta}}$$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta} \cdot \frac{\cos\theta}{\cos\theta + \sin\theta}$$

$$= \frac{1}{\sin\theta}$$

$$= \csc\theta$$

Therefore, the correct answer choice is J.

91. **REVIEW** Which of the following is not equivalent to $\cos\theta$, when $0 < \theta < \frac{\pi}{2}$?

A $\dfrac{\cos\theta}{\cos^2\theta + \sin^2\theta}$

B $\dfrac{1-\sin^2\theta}{\cos\theta}$

C $\cot\theta \sin\theta$

D $\tan\theta \csc\theta$

SOLUTION:

A. $\dfrac{\cos\theta}{\cos^2\theta+\sin^2\theta} = \dfrac{\cos\theta}{1}$ or $\cos\theta$

B. $\dfrac{1-\sin^2\theta}{\cos\theta} = \dfrac{\cos^2\theta}{\cos\theta} = \cos\theta$

C. $\cot\theta\sin\theta = \dfrac{\cos\theta}{\sin\theta} \cdot \sin\theta = \cos\theta$

D. $\tan\theta\csc\theta = \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\sin\theta} = \dfrac{1}{\cos\theta} \neq \cos\theta$

Therefore, the correct answer choice is D.

92. **REVIEW** Which of the following is equivalent to $\sin\theta + \cot\theta\cos\theta$?

F $2\sin\theta$

G $\dfrac{1}{\sin\theta}$

H $\cos^2\theta$

J $\dfrac{\sin\theta + \cos\theta}{\sin^2\theta}$

SOLUTION:

$$\begin{aligned}\sin\theta + \cot\theta\cos\theta &= \sin\theta + \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta}{1} \\ &= \sin\theta + \frac{\cos^2\theta}{\sin\theta} \\ &= \frac{\sin^2\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta} \\ &= \frac{\sin^2\theta+\cos^2\theta}{\sin\theta} \\ &= \frac{1}{\sin\theta}\end{aligned}$$

Therefore, the correct answer choice is G.